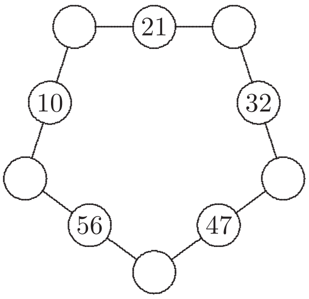
GAUSS STUDENT SAMPLE PROBLEMS: SOLUTIONS

PROBLEM 1

Complete the magic pentagon by putting numbers in the empty circles so that the sum of the three numbers along each of the sides of the pentagon equals 100. Show that there is only one solution to this problem.



SOLUTION 1

Let x be the number that is supposed to be put in the circle which is between the circles with numbers 10 and 21.
Then the number to be put in the circle between the circles containing 21 and 32 has to be equal to $100 - 21 - x = 79 - x$ since the sum of the three numbers along each of the sides of the pentagon equals 100.

$\boxed{1}$

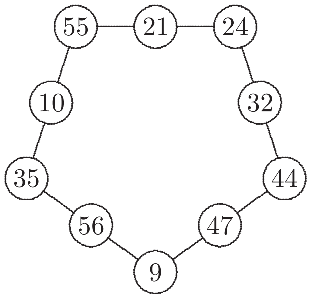
Therefore the number in the circle between the circles containing 32 and 47 has to be equal to $100 - 32 - (79 - x) = x - 11$.
Hence the number in the circle between the circles containing 47 and 56 must be equal to $100 - 47 - (x - 11) = 64 - x$.
Hence the last empty circle must contain $100 - 56 - (64 - x) = x - 20$.

$\boxed{1}$

Thus we see that one of the sides of the pentagon contains circles with numbers 10, x and $x - 20$. Hence we obtain the equation

$$10 + x + x - 20 = 100.$$ 1

Hence $2x = 110$ which implies $x = 55$. Therefore we can calculate the other numbers and obtain the following answer.



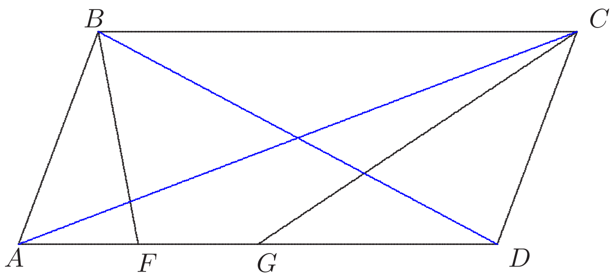
The solution shows that there is only one value of x that satisfies the conditions of the problem and therefore the values of the other four numbers are unique. 1

Note. A computer program or spreadsheet must explain how it has been constructed for full marks (no explanation gets 1 mark - the last). Trial and error or similar solutions which do not show that the answer is the only one gain 1 mark.

PROBLEM 2

$ABCD$ is a parallelogram. G is the midpoint of AD and F is the midpoint of AG. The area of the parallelogram $ABCD$ is 200 cm^2. Find the area of the quadrilateral $BCGF$.

SOLUTION 2



Draw AC and BD.

Since the triangles ACD and CAB are congruent, $|ACD| = |CAB|$.

Hence $|ACD| = \frac{1}{2}|ABCD| = 100 \text{ cm}^2$.

Similarly, $|ABD| = \frac{1}{2}|ABCD| = 100 \text{ cm}^2$. [1]

The triangles GCD and ACD have a common altitude from C and $GD = \frac{1}{2}AD$.

Hence $|GCD| = \frac{1}{2}|ACD| = 50 \text{ cm}^2$. [1]

The triangles ABF and ABD have a common altitude from B and $AF = \frac{1}{4}AD$.

Hence $|ABF| = \frac{1}{4}|ABD| = 25 \text{ cm}^2$. [1]

Therefore

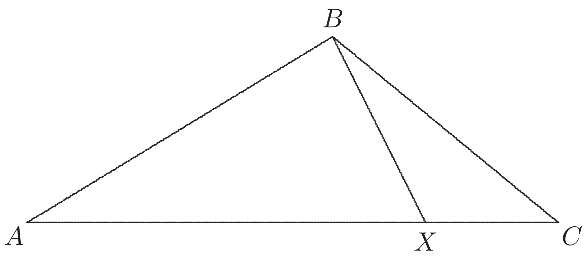
$$\begin{aligned}
|BCGF| &= |ABCD| - |GCD| - |ABF| \\
&= 200 \text{ cm}^2 - 50 \text{ cm}^2 - 25 \text{ cm}^2 = 125 \text{ cm}^2.
\end{aligned}$$

Thus $|BCGF| = 125 \text{ cm}^2$. [1]

PROBLEM 3

In a triangle ABC, X is a point on AC such that $\frac{AX}{XC} = 3$ and $\angle AXB = 70^\circ$. Find the length of AC if $\angle ABC = 110^\circ$ and $BC = 6$ cm.

SOLUTION 3



Since $\angle AXB = 70^\circ$, we have $\angle BXC = 180^\circ - \angle AXB = 180^\circ - 70^\circ = 110^\circ$.
Hence $\angle ABC = \angle BXC$. [1]

Thus in the triangles ABC and BXC,
$\angle ABC = \angle BXC$ and $\angle C$ is their common angle.
Hence ABC and BXC are similar triangles. [1]

Therefore $\frac{AC}{BC} = \frac{BC}{XC}$.
Now let $XC = x$. Then $AX = 3x$ and $AC = AX + XC = 4x$ as $\frac{AX}{XC} = 3$.

Hence $\frac{4x}{6 \text{ cm}} = \frac{6 \text{ cm}}{x}$ as $BC = 6$ cm. [1]
Therefore $4x^2 = 36 \text{ cm}^2$ which yields $x^2 = 9 \text{ cm}^2$.
Hence $x = 3$ cm and $AC = 4x = 12$ cm.
Thus $AC = 12$ cm. [1]

PROBLEM 4

Find all three digit numbers N such that $N - 14$ is divisible by 7, $N - 24$ is divisible by 8 and $N - 36$ is divisible by 9.

SOLUTION 4

Since $N - 14$ is divisible 7 and 14 is divisible by 7, N is also divisible by 7.

$\boxed{1}$

Similarly N is divisible by 8 as $N - 24$ is divisible by 8 and 24 is divisible by 8,
and also N is divisible by 9 as $N - 36$ is divisible by 9 and 36 is divisible by 9. $\boxed{1}$

Thus N is divisible by each of the numbers 7, 8 and 9.
Hence N is divisible by $7 \times 8 \times 9 = 504$. $\boxed{1}$

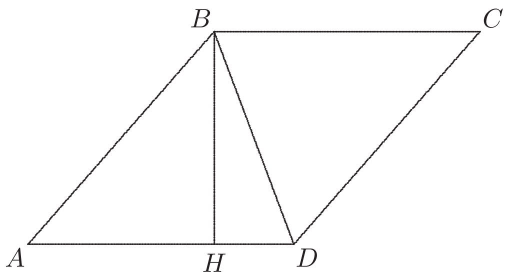
Since there is only one three digit number that is divisible by 504, there is only one number N that satisfies the conditions of the problem, namely $N = 504$. $\boxed{1}$

Note. A computer program or spreadsheet must explain how it has been constructed for full marks (no explanation gets 1 mark - the last). Trial and error or similar solutions which do not show that the answer is the only one gain 1 mark.

PROBLEM 5

In a parallelogram $ABCD$, $AB = 15$ cm and $BC = 14$ cm. The foot H of the perpendicular dropped from B to AD is between A and D. Find the length of the diagonal BD if $BH = 12$ cm.

SOLUTION 5



Since BH is perpendicular to AD, we have $\angle AHB = 90°$.
Therefore by Pythagoras' theorem, we obtain $AH^2 + BH^2 = AB^2$. [1]

Hence $AH^2 = AB^2 - BH^2 = 225\text{ cm}^2 - 144\text{ cm}^2 = 81\text{ cm}^2$.
Therefore $AH = 9$ cm. [1]

Since $ABCD$ is a parallelogram, $AD = BC = 14$ cm.
Hence $HD = AD - AH = 14\text{ cm} - 9\text{ cm} = 5$ cm. [1]

Therefore by Pythagoras' theorem applied to the triangle BHD, we have $BD^2 = BH^2 + HD^2 = 144\text{ cm}^2 + 25\text{ cm}^2 = 169\text{ cm}^2$.
Hence $BD = 13$ cm. [1]

PROBLEM 6

X, Y and Z are positive integers such that $X^2 + Y^2 + Z^2 = 390$. What is the value of $X + Y + Z$? Find all possible solutions.

SOLUTION 6

Since $20^2 = 400$ and $X^2 + Y^2 + Z^2 = 390 < 400$, we see that $X < 20$, $Y < 20$ and $Z < 20$. 1

Set up a spreadsheet with 1 to 19 down a column (X) and across a row (Y). In each cell, calculate $\sqrt{390 - X^2 - Y^2}$. Look for integer values. 1

The following table lists all values for X, Y and Z such that $X^2 + Y^2 + Z^2 = 390$.

X	Y	Z	$X^2+Y^2+Z^2$	$X+Y+Z$
1	10	17	390	28
1	17	10	390	28
2	5	19	390	26
2	19	5	390	26
5	2	19	390	26
5	13	14	390	32
5	14	13	390	32
5	19	2	390	26
10	1	17	390	28
10	11	13	390	34
10	13	11	390	34
10	17	1	390	28
11	10	13	390	34
11	13	10	390	34
13	5	14	390	32
13	10	11	390	34
13	11	10	390	34
13	14	5	390	32
14	5	13	390	32
14	13	5	390	32
17	1	10	390	28
17	10	1	390	28
19	2	5	390	26
19	5	2	390	26

1

Thus the possible values of $X + Y + Z$ are 26, 28, 32 and 34. 1

PROBLEM 7

A carpenter has a long rod of length 4 m 83 cm. He needs to cut this rod into lengths of 161 cm, 23 cm and 7 cm so that he would get at least one rod of each length and have nothing left over. After a while, the carpenter managed to solve this problem. How many rods of each length did he get? Find all possible solutions.

SOLUTION 7

Alternative 1

Let a, b and c be the numbers of rods of lengths 161 cm, 23 cm and 7 cm respectively that were obtained by the carpenter.
Then the following equation holds:

$$161a + 23b + 7c = 483.$$ 1

Since $483 = 23 \times 7 \times 3$ and $161 = 23 \times 7$, we can see that each of $161a$, $23b$ and 483 is divisible by 23.
Hence $7c$ is divisible by 23 which implies that c is divisible by 23.

Similarly since each of $161a$, $7c$ and 483 is divisible by 7, $23b$ must be divisible by 7 which means that b is divisible by 7.

Therefore we have $b = 7m$ and $c = 23n$ as b is divisible by 7 and c is divisible by 23.
Hence $161a + 161m + 161n = 483$ which yields $a + m + n = 3$. 1
Since none of a, m and c is allowed to be 0, the only solution is $a = 1$, $m = 1$ and $n = 1$.
Hence $a = 1$, $b = 7$ and $c = 23$. 1

Thus the carpenter obtained one rod of length 161 cm, seven rods of length 23 cm and twenty three rods of length 7 cm, and this is the only answer to the problem. 1

Alternative 2

Let a, b and c be the numbers of rods of lengths 161 cm, 23 cm and 7 cm respectively that were obtained by the carpenter.

Since $161 \times 3 = 483$ and the carpenter got at least one rod of each length, we see that either $a = 2$ or $a = 1$. 1

If $a = 2$, then $23b + 7c = 483 - 161 \times 2 = 161$.
Since both 161 and $7c$ are divisible by 7, $23b$ must be divisible by 7.
Hence $b \geq 7$ and therefore $23b \geq 161$ which is not possible as $c > 0$. 1

If $a = 1$, then $23b + 7c = 322$.
Since $322 = 23 \times 14$, we have $1 \leq b \leq 13$.
We now check all possible values of b and come to the conclusion that only $b = 7$, $a = 23$ satisfies all the restraints. 1

Thus the carpenter obtained one rod of length 161 cm, seven rods of length 23 cm and twenty three rods of length 7 cm, and this is the only answer to the problem. 1

Note. A computer program or spreadsheet must explain how it has been constructed for full marks (no explanation gets 1 mark - the last). Trial and error or similar solutions which do not show that the answer is the only one gain 1 mark.

PROBLEM 8

Points A and B are on the banks of a river. It takes a motorboat 1 hour to go down-stream from A to B and 1 hour 30 min to go up-stream from B to A. How long would it take the motorboat to go from C to D and back, if C and D are on the shores of a lake and the distance between C and D is the same as that between A and B?

SOLUTION 8

Let the distance between A and B be d km, the speed of the motorboat in the lake be u km/h and the speed of the current of the river v km/h.

Then we have

$$u + v = \frac{d}{1} \qquad \boxed{1}$$

and

$$u - v = \frac{d}{1.5} \qquad \boxed{1}$$

as 1 hour 30 min is 1.5 hours.

Hence

$$\begin{aligned} 2u &= (u+v) + (u-v) \\ &= \frac{d}{1} + \frac{d}{1.5} \\ &= \frac{5d}{3}. \end{aligned}$$

Thus $2u = \frac{5d}{3}$ which implies $u = \frac{5d}{6}$. $\boxed{1}$

Therefore the trip from C to D and back takes the motorboat

$$2d \div \frac{5d}{6} = \frac{12d}{5d} = 2.4$$

hours.

Thus the answer is 2 hours and 24 minutes. $\boxed{1}$

PROBLEM 9

The Executive Director of the "Smart Solutions" company wishes to employ 9 new staff members. There are 14 applicants: 4 mathematicians and 10 accountants. In how many ways can the Executive Director select 9 people if he wants to employ at least one mathematician?

SOLUTION 9

Alternative 1

The group of the nine successful applicants consists of either one mathematician and eight accountants or two mathematicians and seven accountants or three mathematicians and six accountants or four mathematicians and five accountants. $\boxed{1}$

There are $\binom{4}{1} \times \binom{10}{8} = 4 \times 45 = 180$ ways to select a group consisting of one mathematician and eight accountants, $\boxed{1}$

there are $\binom{4}{2} \times \binom{10}{7} = 6 \times 120 = 720$ ways to select a group consisting of two mathematicians and seven accountants,

there are $\binom{4}{3} \times \binom{10}{6} = 4 \times 210 = 840$ ways to select a group consisting of three mathematicians and six accountants,

and there are $\binom{4}{4} \times \binom{10}{5} = 1 \times 252 = 252$ ways to select a group consisting of four mathematicians and five accountants. $\boxed{1}$

Therefore, there are $180+720+840+252 = 1992$ ways to select 9 people. $\boxed{1}$

Alternative 2

Let M be the number of ways to select 9 people without knowing how many of them are mathematicians.

Let N be the number of ways to select 9 people so that all of them are accountants.

Then the number of ways the executive director can select 9 people so that at least one of them is a mathematician is $M - N$. $\boxed{1}$

There are $\binom{14}{9} = 2002$ ways to select 9 people without knowing how many of them are mathematicians. Hence $M = 2002$. $\boxed{1}$

There are $\binom{10}{9} = 10$ ways to select a group consisting of accountants only. Hence $N = 10$. $\boxed{1}$

Therefore, there are $M - N = 2002 - 10 = 1992$ ways to select 9 people so that at least one of them is a mathematician. $\boxed{1}$

Note. A computer program solution must explain how it has been constructed for full marks (no explanation gets 1 mark - the last).

PROBLEM 10

Show that $29 \times 20^{99} + 71 \times 56^{64} + 24 \times 37^{55}$ is divisible by 19.

SOLUTION 10

We have $20 \equiv 1 \pmod{19}$.
Hence $20^{99} \equiv 1^{99} \equiv 1 \pmod{19}$.
Therefore $29 \times 20^{99} \equiv 29 \times 1 \equiv 29 \equiv 10 \pmod{19}$. $\boxed{1}$

Also we can see that $56 \equiv -1 \pmod{19}$.
Hence $56^{64} \equiv (-1)^{64} \equiv 1 \pmod{19}$.
Therefore $71 \times 56^{64} \equiv 71 \times 1 \equiv 71 \equiv 14 \pmod{19}$. $\boxed{1}$

Finally, $37 \equiv -1 \pmod{19}$.
Hence $37^{55} \equiv (-1)^{55} \equiv -1 \equiv 18 \pmod{19}$.
Therefore $24 \times 37^{55} \equiv 24 \times 18 \equiv 14 \pmod{19}$. $\boxed{1}$

Hence

$$29 \times 20^{99} + 71 \times 56^{64} + 24 \times 37^{55} \equiv 10 + 14 + 14 \equiv 38 \equiv 0 \pmod{19}$$

which means that $29 \times 20^{99} + 71 \times 56^{64} + 24 \times 37^{55}$ is divisible by 19.
$\boxed{1}$

PROBLEM 11

For the Pan Australian Games the three states at the top of the medals table (Victoria, S.A. and W.A.) had won a total of 186 medals. Victoria had won the most gold medals and S.A. had as many gold medals as bronze medals. Victoria and S.A. won the same number of silver medals. W.A. had two more silver medals than bronze medals and their gold medals numbered one more than Victoria's bronze medals. Victoria had as many gold medals as the bronze medals of S.A. and W.A. combined and this number was also three quarters of the total number of medals won by S.A. The number of gold medals won by the three states was one less than the total number of medals won by Victoria. How many of each medal did each state win?

SOLUTION 11

Let a be the number of silver medals won by Victoria.
Let b be the number of gold medals won by S.A.
Let c be the number of bronze medals won by W.A.
Let d be the number of bronze medals won by Victoria.

Then a table can be constructed as shown:

	Gold	Silver	Bronze
Victoria	$b+c$	a	d
W.A.	$d+1$	$c+2$	c
S.A.	b	a	b

1

Also we have

$$\begin{aligned} \text{Sum of medals} &= 186 \\ b+c &= \frac{3(a+2b)}{4} \\ (b+c)+(d+1)+b &= (b+c)+a+d-1. \end{aligned}$$

1

Simplifying these become

$$\begin{aligned} 2a+3b+3c+2d &= 183 \\ 4c &= 3a+2b \\ b &= a-2. \end{aligned}$$

Solving for c and d in terms of a

$$\begin{aligned} c &= \frac{5a}{4} - 1 \\ d &= 96 - \frac{35a}{8}. \end{aligned}$$

Therefore a is divisible by 8 and must be less than 22 for d to be positive.
Hence $a = 8$ or 16.
Hence $(a, b, c, d) = (8, 6, 9, 61)$ or $(16, 14, 19, 26)$. 1

Since Victoria has the most gold medals, $b + c > d + 1$.
Therefore the latter solution is the required one and the table becomes

	Gold	Silver	Bronze
Victoria	33	16	26
W.A.	27	21	19
S.A.	14	16	14

1

PROBLEM 12

A number is written on a whiteboard. Jim writes 221 to the right of this number thus obtaining a new number which turns out to be a multiple of the original one. What is the original number? Find all possible solutions.

SOLUTION 12

Let the original number be X.
Then the new number on the whiteboard is $1000X + 221$. $\boxed{1}$
Hence we obtain the equation $1000X + 221 = kX$ where k is a positive integer.
Therefore $(k - 1000)X = 221$. $\boxed{1}$

Since $221 = 13 \times 17$, we come to the conclusion that either $X = 1$ or $X = 13$ or $X = 17$ or $X = 221$. $\boxed{1}$

Checking shows that each of these four values for X suits the data.
Thus the original number is one of the numbers 1, 13, 17 and 221. $\boxed{1}$