PÓLYA NECKLACE THEOREM

The Australian Mathematics Trust web page on Pólya describes the version of this theorem when the number of beads n is prime and necklaces which become the same under rotation are considered equal.

The more general form of this requires first an understanding of the following two terms:

Relatively prime: Two numbers a and b are said to be relatively prime if their greatest common divisor is 1.

Examples:

1. 1 is relatively prime to all integers including itself.
2. 14 is relatively prime to 15.
3. 6 and 9 are not relatively prime.

Euler's function: Euler's function $\phi(a)$ can be defined as the number of integers from 1 to a relatively prime to a.

Examples:

1. $\phi(1) = 1$.
2. If a is prime, $\phi(a) = a - 1$ since a is relatively prime to all numbers from 1 to $a - 1$ but not itself.
3. If a is composite we need to check all the the numbers from 1 to $a - 1$. For example, if $a = 4$, a is relatively prime to 1 and 3 but not 2, so $\phi(4) = 2$.

NECKLACE THEOREM version 1

We now state the Necklace Theorem for the case of rotations being considered equal. Here the number of different necklaces, if there are to be n beads and an infinite supply of each of k colours, is given by

$$N = \left(\frac{1}{n}\right) \sum_d \phi(d) k^{\left(\frac{n}{d}\right)}$$

where the sum is over all divisors d of n and ϕ is Euler's function.

Examples:

1. If n is prime the only two terms in the sum are for $d = 1$ and $d = n$. From above, then

$$\begin{aligned} N &= \left(\frac{1}{n}\right) \left(\phi(1)k^n + \phi(n)k^1\right) \\ &= \left(\frac{1}{n}\right) (k^n + (n-1)k) \\ &= k + \frac{k^n - k}{n}. \end{aligned}$$

Note that this is the case on the Australian Mathematics Trust web site and for $n = 5$ and $k = 3$ this gives

$$N = 3 + \frac{3^5 - 3}{5} = 3 + \frac{243 - 3}{5} = 3 + \frac{240}{5} = 3 + 48 = 51.$$

2. Now consider a case in which n is composite, say $n = 4$. Here the sum needs to allow for cases $d = 1$, $d = 2$ and $d = 4$. Since $\phi(1) = 1$, $\phi(2) = 1$ and $\phi(4) = 2$ this gives

$$N = \frac{1}{4}(1.k^4 + 1.k^2 + 2.k^1) = \frac{1}{4}(k^4 + k^2 + 2k).$$

If the value of k is 3 this gives

$$N = \frac{1}{4}(3^4 + 3^2 + 2.3) = \frac{1}{4}(81 + 9 + 6) = \frac{96}{4} = 24.$$

NECKLACE THEOREM version 2

We now go on to state the Necklace Theorem for the case of necklaces where rotations are considered the same and reflections are considered the same. Here the number of different necklaces N, if there are to be n beads and at most k colours, is given by two general formulae, one for even n and one for odd n. If n is odd we have

$$N = \frac{1}{2n}\left(\sum_d \phi(d) k^{\frac{n}{d}}\right) + \frac{1}{2} k^{\frac{n+1}{2}}$$

and

$$N = \frac{1}{2n}\left(\sum_d \phi(d) k^{\frac{n}{d}}\right) + \frac{1}{4}\left(k^{\frac{(n+2)}{2}} + k^{\frac{n}{2}}\right)$$

if n is even.

For odd prime n, this reduces to

$$N = \frac{1}{2n}\left(k^n - k\right) + \frac{1}{2}\left(k^{\frac{n+1}{2}} + k\right).$$

For example, for $n = 5$ and $k = 3$ we get

$$N = \frac{1}{10}\left(3^5 - 3\right) + \frac{1}{2}\left(3^3 + 3\right) = 24 + 15 = 39.$$

(Recall this answer was 51 in version 1, when reflections could be different.)

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